```
> restart
```

D-Algebraic Functions

This worksheet accompagnies the paper "D-Algebraic Functions" by Rida Ait El Manssour, Anna-Laura Sattelberger, and Bertrand Teguia Tabuguia. Our Maple package **NLDE** uses the Maple packages

- PolynomialIdeals
- Groebner

for elimination with Gröbner bases. The source file for this version of the package can be found here, and the up-to-date version is available here. Those interested in trying the computations of this file in a Jupyter notebook should download the notebook attached to the MathRepo webpage at https://mathrepo.mis.mpg.de/DAlgebraicFunctions

In order to be able to use NLDE, one can just put the file NLDE.mla in the same directory with the notebook or the Maple worksheet.

We start by setting the (current) working directory as a directory for libraries.

```
> libname:=currentdir(),libname:
```

Then we load the NLDE package. This worksheet is concerns with 4 procedures.

```
> with(NLDE,SysToMinDiffPoly,arithmeticDalg,composeDalg,unaryDalg)
```

$$[SysToMinDiffPoly, arithmeticDalg, composeDalg, unaryDalg] \quad (1)$$

Example 2.3

From the linear ODE of $\sin(x)$ one derives an ADE satisfied by its reciprocal using *unaryDalg* as below.

The equation for $\sin(x)$ can be computed using *DEtools:-FindODE*.

```
> ODE:=DEtools:-FindODE(sin(x),y(x))
```

$$ODE := y(x) + \frac{d^2}{dx^2} y(x) \quad (2)$$

Syntax of *unaryDalg*: *unaryDalg*(ADEin,y(x),z=r(x,y)), where ADEin is an algebraic differential equation (so contains "="), y(x) is its dependent variable, and x is its independent variable. z is the name of dependent variable for the output (like y for y(x)), and r is a rational expression in x and y. The output is an ADE satisfies by all r(x,f(x)), where f(x) satisfies the ADEin.

Hence the ADE for the reciprocal $\sin(x)$

```
> unaryDalg(ODE=0,y(x),z=1/y)
```

$$-z(x)^2 + z(x)\left(\frac{d^2}{dx^2}z(x)\right) - 2\left(\frac{d}{dx}z(x)\right)^2 = 0 \tag{3}$$

Example 2.9

We now consider the Painlevé transcendent of type I that fullfils the ADE:

$$\frac{d^2}{dx^2}y(x) = 6\,y(x)^2 + x$$

All squares of solutions to that ADE satisfy the following ADE

```
> unaryDalg(diff(y(x),x,x)=6*y(x)^2+x,y(x),z=y^2)
```

$$16\,x^2 z(x)^3 + 192\,x\,z(x)^4 + 576\,z(x)^5 - 4\,z(x)^2\left(\frac{d^2}{dx^2}z(x)\right)^2 + 4\,z(x)\left(\frac{d}{dx}z(x)\right)^2\left(\frac{d^2}{dx^2}z(x)\right) - \left(\frac{d}{dx}z(x)\right)^4 = 0 \tag{4}$$

Example 4.4

Syntax of *composeDalg*: *composeDalg*([ADE1,ADE2],[y(x),z(x)],w(x)),

where ADE1 and ADE2 are two algebraic differential equations of the dependent variable y(x) and z(x), resp. w(x) is the dependent variable for the output. The latter is an ADE satisfies by all f(g(x)) where f(x) satisfies ADE1, and g(x) satisfies ADE2.

```
> ADE1:=diff(y(x),x)-y(x)=0
```

$$ADE1 := \frac{d}{dx}y(x) - y(x) = 0 \tag{5}$$

```
> ADE2:=z(x)^2+2*diff(z(x),x)=0
```

$$ADE2 := z(x)^2 + 2\,\frac{d}{dx}z(x) = 0 \tag{6}$$

```
> composeDalg([ADE1,ADE2],[y(x),z(x)],w(x))
```

$$w(x)^2 \left(\frac{d^2}{dx^2} w(x)\right)^2 + 2\, w(x) \left(\frac{d}{dx} w(x)\right)^3 - 2\, w(x) \left(\frac{d}{dx} w(x)\right)^2 \left(\frac{d^2}{dx^2} w(x)\right) + \left(\frac{d}{dx} w(x)\right)^4 = 0 \tag{7}$$

Example 4.5

```
> ADE1:=diff(y(x),x,x)+y(x)=0
```

$$ADE1 := y(x) + \frac{d^2}{dx^2} y(x) = 0 \tag{8}$$

```
> ADE2:=diff(z(x),x)-x*z(x)=0
```

$$ADE2 := \frac{d}{dx} z(x) - x\, z(x) = 0 \tag{9}$$

```
> composeDalg([ADE1,ADE2],[y(x),z(x)],w(x))
```

$$\left(2\, x^4 + 3\, x^2 + 3\right) w(x) \left(\frac{d}{dx} w(x)\right) + \left(-3\, x^3 - 3\, x\right) w(x) \left(\frac{d^2}{dx^2} w(x)\right) + \left(\frac{d^3}{dx^3} w(x)\right) x^2\, w(x) + \left(\frac{d}{dx} w(x)\right)^2 \left(x^3 + x\right) - \left(\frac{d}{dx} w(x)\right) x^2 \left(\frac{d^2}{dx^2} w(x)\right) = 0 \tag{10}$$

Example 5.1

Syntax of *arithmeticDalg*: *arithmeticDalg*([ADE1,...,ADEn],[y1(x),...,yn(x)],w=r(x,y1,...,yn)),

to be understood as for *unaryDalg*, but here we can deal with n-ary (n operands) operations with D-algebraic functions.

```
> ADE1:=diff(y(x),x)^3+y(x)+1=0
```

$$ADE1 := \left(\frac{d}{dx} y(x)\right)^3 + y(x) + 1 = 0 \tag{11}$$

```
> ADE2:=diff(z(x),x)^2-z(x)-1=0
```

$$ADE2 := \left(\frac{d}{dx} z(x)\right)^2 - z(x) - 1 = 0 \tag{12}$$

```
> arithmeticDalg([ADE1,ADE2],[y(x),z(x)],w=y+z)
```

$$-24 \left(\frac{d^2}{dx^2} w(x)\right)^3 + 36 \left(\frac{d^2}{dx^2} w(x)\right)^2 - 18\, \frac{d^2}{dx^2} w(x) + 8\, \frac{d^3}{dx^3} w(x) + 3 = 0 \tag{13}$$

Remark 5.2 and Remark 5.3

```
> ADE1:=diff(y(x),x,x)*y(x)-diff(y(x),x)^2=0
```
$$ADE1 := \left(\frac{d^2}{dx^2}\,y(x)\right)y(x) - \left(\frac{d}{dx}\,y(x)\right)^2 = 0 \tag{14}$$

```
> ADE2:=diff(z(x),x)^2+z(x)^2+1=0
```
$$ADE2 := \left(\frac{d}{dx}\,z(x)\right)^2 + z(x)^2 + 1 = 0 \tag{15}$$

```
> arithmeticDalg([ADE1,ADE2],[y(x),z(x)],w=y+z)
```
$$w(x)\left(\frac{d^2}{dx^2}\,w(x)\right) + w(x)\left(\frac{d^4}{dx^4}\,w(x)\right) - \left(\frac{d}{dx}\,w(x)\right)^2 - 2\left(\frac{d}{dx}\,w(x)\right)\left(\frac{d^3}{dx^3}\,w(x)\right) + \left(\frac{d^2}{dx^2}\,w(x)\right)^2 + \left(\frac{d^2}{dx^2}\,w(x)\right)\left(\frac{d^4}{dx^4}\,w(x)\right) - \left(\frac{d^3}{dx^3}\,w(x)\right)^2 = 0 \tag{16}$$

One can verify that $f \pm I$ is not a solution of the previous output ADE, where f is a solution of ADE1, eg: $f(x) = \exp(x)$, and I, the imaginary number, is a constant solution of ADE2.
This special case can be treated separately as shown below.

```
> unaryDalg(ADE1,y(x),w=y+I)
```
$$-I\left(\frac{d^2}{dx^2}\,w(x)\right) + w(x)\left(\frac{d^2}{dx^2}\,w(x)\right) - \left(\frac{d}{dx}\,w(x)\right)^2 = 0 \tag{17}$$

An algebraic equation is seen as a zeroth-order differential equation.

```
> arithmeticDalg([ADE1,z(x)^2+1=0],[y(x),z(x)],w=y+z)
```
$$\left(\frac{d}{dx}\,w(x)\right)\left(\frac{d^3}{dx^3}\,w(x)\right) - \left(\frac{d^2}{dx^2}\,w(x)\right)^2 = 0 \tag{18}$$

Example 5.4 (Elliptic functions)

We invite the reader to try the computations to see the outputs. Some outputs are quite big.

> $ADEwp := diff(p(z), z)^2 = 4 \cdot p(z)^3 - g2 \cdot p(z) - g3$

> $\mathrm{Kappa} := \dfrac{-3\,a1\,a13\,a24\,p + a1^2\,s\_1 - 2\,a1\,s\_2 + 3\,s\_3}{-3\,a13\,a24\,p + 3\,a1^2 - 2\,a1\,s\_1 + s\_2}$

> $s\_1 := a2 + a3 + a4;\ s\_2 := a2 \cdot a3 + a2 \cdot a4 + a3 \cdot a4;\ s\_3 := a2 \cdot a3 \cdot a4$

> $a13 := a1 - a3;\ a24 := a2 - a4$

>

We use our algorithm to find the ADE satisfied by kappa

$$> \ ADEkappa := unaryDalg(ADEwp, p(z), \mathrm{kappa} = \mathrm{Kappa})$$

We rearrange it for identification of the coefficients.

$$> \ PolyADE := lhs\left(subs\left(\kappa(z) = X, subs\left(diff\left(\kappa(z), z\right) = Y, ADEkappa\right)\right)\right) :$$
$$> \ diffcoeff := factor(coeff(PolyADE, Y, 2))$$
$$> \ coeffX4 := factor(coeff(PolyADE, X, 4)) :$$
$$> \ coeffX3 := factor(coeff(PolyADE, X, 3)) :$$
$$> \ coeffX2 := factor(coeff(PolyADE, X, 2)) :$$
$$> \ coeffX1 := factor(coeff(PolyADE, X)) :$$
$$> \ coeffX0 := factor(subs([Y = 0, X = 0], PolyADE)) :$$
$$> \ constfact := 4 \cdot 27 \cdot (a1 - a4)^2 (a1 - a3)^2 (a1 - a2)^2$$
$$> \ diffcoeff := \frac{diffcoeff}{constfact}$$
$$> \ coeffX4 := factor\left(\frac{coeffX4}{constfact}\right)$$
$$> \ coeffX3 := factor\left(\frac{coeffX3}{constfact}\right)$$
$$> \ coeffX2 := factor\left(\frac{coeffX2}{constfact}\right)$$
$$> \ coeffX1 := factor\left(\frac{coeffX1}{constfact}\right)$$
$$> \ coeffX0 := factor\left(\frac{coeffX0}{constfact}\right)$$

The expected equation and its coefficients

$$>$$
$$> \ Eq := \left(c4 \frac{\mathrm{d}}{\mathrm{d}z} \kappa(z)\right)^2 = (\kappa(z) - a1) \cdot (\kappa(z) - a2) \cdot (\kappa(z) - a3) \cdot (\kappa(z) - a4)$$
$$> \ EX := collect(expand(rhs(Eq)), \kappa(z), distributed)$$

We solve the system

$$> \ Solg2g3 := solve\left(\left[coeffX4 = 1, coeffX3 = coeff\left(EX, \kappa(z), 3\right), coeffX2 = coeff\left(EX, \kappa(z), 2\right), coeffX1 = coeff\left(EX, \kappa(z), 1\right), coeffX0 = coeff\left(EX, \kappa(z), 0\right)\right], [g2, g3]\right) :$$
$$> \ Subg2g3 := Solg2g3[1] :$$
$$> \ map(factor, Subg2g3)$$
$$>$$

Verification (recovering the equation from the expressions of g2 and g3)

> $ADEwp$

> $ADEwp\_g2g3 := subs(Subg2g3, ADEwp)$

> $ADEkappa\_g2g3 := NLDE{:}\text{-}unaryDalg(ADEwp\_g2g3, p(z), \mathrm{kappa} = \mathrm{Kappa})$

> $map(factor, lhs(ADEkappa\_g2g3))$

> $(a2 - a4)\,(a1 - a3)\left(\frac{d}{dz}\,\kappa(z)\right)^2 = factor(subs(diff(\mathrm{kappa}(z), z) = 0, \%))$

>

Example 5.5 (SIR model)

Syntax of *SystoMinDiffPoly*: *SystoMinDiffPoly*([dy1,...,dyn],r(x,y1,...yn),[y1,...yn],w(x)),

where dy1,...,dyn, are the derivatives of y1,...,yn, in terms of y1,...,yn, given rationally. x is the independent variable, and w is the name of the dependent variable for the output differential equation. This is also called input-output equation.

```
> timing,p:=CodeTools:-CPUTime(SysToMinDiffPoly([-beta*S*T-delta*S
  + mu, beta*S*
  T - gamma*T + nu, delta*S+gamma*T],R,[S,T,R],w(t)))
```

$$\begin{aligned}
timing, p := 0.453,\ & \left(\beta^2\delta^2 - \beta^2\gamma^2\right)\left(\frac{d}{dt}\,w(t)\right)^3\left(\frac{d^2}{dt^2}\,w(t)\right) + \left(\beta^2\delta - \beta^2\gamma\right)\left(\frac{d}{dt}\,w(t)\right)^2\left(\frac{d^2}{dt^2}\,w(t)\right)^2 + \left(-\beta^2\delta^2\mu - \beta^2\delta^2\nu + \beta^2\gamma^2\mu + \beta^2\gamma^2\nu + 2\,\beta\,\delta^3\gamma + 8\,\beta\,\delta^2\gamma^2\right. \\
& \left. + 2\,\beta\,\delta\,\gamma^3\right)\left(\frac{d}{dt}\,w(t)\right)^2\left(\frac{d^2}{dt^2}\,w(t)\right) + \left(6\,\beta\,\delta^2\gamma + 6\,\beta\,\delta\,\gamma^2\right)\left(\frac{d^2}{dt^2}\,w(t)\right)^2\left(\frac{d}{dt}\,w(t)\right) \\
& + \left(-6\,\beta\,\delta^3\gamma\mu - 2\,\beta\,\delta^3\gamma\nu - 16\,\beta\,\delta^2\gamma^2\mu - 16\,\beta\,\delta^2\gamma^2\nu - 2\,\beta\,\delta\,\gamma^3\mu - 6\,\beta\,\delta\,\gamma^3\nu + \delta^4\gamma^2\right. \\
& \left. - \delta^2\gamma^4\right)\left(\frac{d}{dt}\,w(t)\right)\left(\frac{d^2}{dt^2}\,w(t)\right) + \left(-\delta^3\gamma + \delta\,\gamma^3\right)\left(\frac{d^3}{dt^3}\,w(t)\right)\left(\frac{d^2}{dt^2}\,w(t)\right) + \delta^4\gamma^3\mu^2 \\
& + \delta^4\gamma^3\nu^2 - \delta^3\gamma^4\mu^2 - \delta^3\gamma^4\nu^2 - 8\,\beta\,\delta^3\gamma^2\mu^2\nu - 4\,\beta\,\delta^3\gamma^2\mu\,\nu^2 - 4\,\beta\,\delta^2\gamma^3\mu^2\nu - 8\,\beta\,\delta^2\gamma^3\mu\,\nu^2 \\
& + \left(\beta\,\delta^2 + 2\,\beta\,\delta\,\gamma + \beta\,\gamma^2\right)\left(\frac{d^2}{dt^2}\,w(t)\right)^3 + \left(-\beta\,\delta^3\mu - 6\,\beta\,\delta^2\gamma\mu - 5\,\beta\,\delta^2\gamma\nu - 5\,\beta\,\delta\,\gamma^2\mu\right. \\
& \left. - 6\,\beta\,\delta\,\gamma^2\nu - \beta\,\gamma^3\nu\right)\left(\frac{d^2}{dt^2}\,w(t)\right)^2 + \left(4\,\beta\,\delta^3\gamma\mu^2 + 4\,\beta\,\delta^3\gamma\mu\,\nu + 8\,\beta\,\delta^2\gamma^2\mu^2\right. \\
& + 16\,\beta\,\delta^2\gamma^2\mu\,\nu + 8\,\beta\,\delta^2\gamma^2\nu^2 + 4\,\beta\,\delta\,\gamma^3\mu\,\nu + 4\,\beta\,\delta\,\gamma^3\nu^2 - \delta^4\gamma^2\mu - \delta^4\gamma^2\nu + \delta^2\gamma^4\mu
\end{aligned} \qquad \textbf{(19)}$$

$$+\delta^2\gamma^4\nu)\left(\frac{d^2}{dt^2}w(t)\right)+(-\delta^2\gamma+\delta\gamma^2)\left(\frac{d^3}{dt^3}w(t)\right)^2+(\beta^2\delta^2\gamma-\beta^2\delta\gamma^2)\left(\frac{d}{dt}w(t)\right)^4$$
$$+(-2\beta^2\delta^2\gamma\mu-2\beta^2\delta^2\gamma\nu+2\beta^2\delta\gamma^2\mu+2\beta^2\delta\gamma^2\nu+2\beta\delta^3\gamma^2+2\beta\delta^2\gamma^3)\left(\frac{d}{dt}w(t)\right)^3$$
$$+(\beta^2\delta^2\gamma\mu^2+2\beta^2\delta^2\gamma\mu\nu+\beta^2\delta^2\gamma\nu^2-\beta^2\delta\gamma^2\mu^2-2\beta^2\delta\gamma^2\mu\nu-\beta^2\delta\gamma^2\nu^2-8\beta\delta^3\gamma^2\mu$$
$$-4\beta\delta^3\gamma^2\nu-4\beta\delta^2\gamma^3\mu-8\beta\delta^2\gamma^3\nu+\delta^4\gamma^3-\delta^3\gamma^4)\left(\frac{d}{dt}w(t)\right)^2+(10\beta\delta^3\gamma^2\mu^2$$
$$+12\beta\delta^3\gamma^2\mu\nu+2\beta\delta^3\gamma^2\nu^2+2\beta\delta^2\gamma^3\mu^2+12\beta\delta^2\gamma^3\mu\nu+10\beta\delta^2\gamma^3\nu^2-2\delta^4\gamma^3\mu$$
$$-2\delta^4\gamma^3\nu+2\delta^3\gamma^4\mu+2\delta^3\gamma^4\nu)\left(\frac{d}{dt}w(t)\right)+(-\beta\delta^2+2\beta\delta\gamma-\beta\gamma^2)\left(\frac{d^2}{dt^2}\right.$$
$$\left.w(t)\right)\left(\frac{d}{dt}w(t)\right)\left(\frac{d^3}{dt^3}w(t)\right)-4\beta\delta^3\gamma^2\mu^3-4\beta\delta^2\gamma^3\nu^3+2\delta^4\gamma^3\mu\nu-2\delta^3\gamma^4\mu\nu=0$$

```
> PDEtools:-difforder(p,t)
```

$$3 \tag{20}$$

Bonus material: checking the validity of the outputs

Composition, addition, multiplication, division, exponentiation.

We here demonstrate how one can verify the correctness of the outputs when the solutions of the input ADEs can be found. Given two algebraic differential equations ADE1 and ADE2, such that f fulfils ADE1 and g fulfills ADE2, the output ADE is valid if it vanishes at $\alpha(f,g)=f\alpha g$, where alpha (e.g.: $+,\cdot,/,\circ$) is the operation used to compute the output ADE.

```
> ADE1:=(1 + x)*diff(y(x), x)=1
```

$$ADE1 := (1+x)\left(\frac{d}{dx}y(x)\right)=1 \tag{21}$$

We solve ADEs using Maple's dsolve command.

```
> dsolve(ADE1,y(x))
```

$$y(x)=\ln(1+x)+\_C1 \tag{22}$$

```
> ADE2:=-z(x)^2 - 2*diff(z(x), x)^2 + diff(z(x), x, x)*z(x) = 0
```

$$ADE2 := -z(x)^2+z(x)\left(\frac{d^2}{dx^2}z(x)\right)-2\left(\frac{d}{dx}z(x)\right)^2=0 \tag{23}$$

```
> dsolve(ADE2,z(x))
```

(24)

$$z(x) = \frac{1}{\_C1 \sin(x) - \_C2 \cos(x)} \tag{24}$$

We will verify our result with $\frac{1}{\sin(x)}$ from the solutions of ADE2 and $\log(1+x)$ from those of ADE1

```
> ADE3:=composeDalg([ADE1,ADE2],[y(x),z(x)],w(x))
```

$$ADE3 := -\left(\frac{d}{dx} w(x)\right)^7 - 3\left(\frac{d}{dx} w(x)\right)^5 \left(\frac{d^2}{dx^2} w(x)\right) + 7\left(\frac{d}{dx} w(x)\right)^5 + \left(\frac{d}{dx} w(x)\right)^4 \left(\frac{d^3}{dx^3} w(x)\right) - 3\left(\frac{d}{dx} w(x)\right)^3 \left(\frac{d^2}{dx^2} w(x)\right)^2 + 6\left(\frac{d}{dx} w(x)\right)^3 \left(\frac{d^2}{dx^2} w(x)\right) + 6\left(\frac{d}{dx} w(x)\right)^2 \left(\frac{d^2}{dx^2} w(x)\right)\left(\frac{d^3}{dx^3} w(x)\right) - 9\left(\frac{d}{dx} w(x)\right)\left(\frac{d^2}{dx^2} w(x)\right)^3 + 8\left(\frac{d}{dx} w(x)\right)^3 - 8\left(\frac{d}{dx} w(x)\right)^2 \left(\frac{d^3}{dx^3} w(x)\right) + 15\left(\frac{d}{dx} w(x)\right)\left(\frac{d^2}{dx^2} w(x)\right)^2 + 2\left(\frac{d}{dx} w(x)\right)\left(\frac{d^3}{dx^3} w(x)\right)^2 - 3\left(\frac{d^2}{dx^2} w(x)\right)^2 \left(\frac{d^3}{dx^3} w(x)\right) = 0 \tag{25}$$

```
> simplify(eval(ADE3,w(x)=log(1+1/sin(x))))
```

$$0 = 0 \tag{26}$$

```
> ADE4:=arithmeticDalg([ADE1,ADE2],[y(x),z(x)],w=y+z)
```

$$ADE4 := \left(\frac{d}{dx} w(x)\right)^3 (8x^7 + 56x^6 + 168x^5 + 280x^4 + 280x^3 + 168x^2 + 56x + 8) + \left(\frac{d}{dx} w(x)\right)^2 \left(\frac{d^3}{dx^3} w(x)\right)(-8x^7 - 56x^6 - 168x^5 - 280x^4 - 280x^3 - 168x^2 - 56x - 8) + \left(\frac{d}{dx} w(x)\right)^2 (-24x^6 - 144x^5 - 344x^4 - 416x^3 - 264x^2 - 80x - 8) + \left(\frac{d}{dx} w(x)\right)\left(\frac{d^2}{dx^2} w(x)\right)^2 (15x^7 + 105x^6 + 315x^5 + 525x^4 + 525x^3 + 315x^2 + 105x + 15) + \left(\frac{d}{dx} w(x)\right)\left(\frac{d^2}{dx^2} w(x)\right)(30x^5 + 150x^4 + 300x^3 + 300x^2 + 150x + 30) + \left(\frac{d}{dx} w(x)\right)\left(\frac{d^3}{dx^3} w(x)\right)^2 (2x^7 + 14x^6 + 42x^5 + 70x^4 + 70x^3 + 42x^2 + 14x + 2) + \left(\frac{d}{dx} w(x)\right)\left(\frac{d^3}{dx^3} w(x)\right)(16x^6 + 96x^5 + 232x^4 + 288x^3 + 192x^2 + 64x + 8) + \left(\frac{d}{dx}\right.$$

$$w(x)\Big)\left(24x^5+120x^4+223x^3+189x^2+77x+15\right)+\left(\frac{d^2}{dx^2}w(x)\right)^2\left(\frac{d^3}{dx^3}w(x)\right)\left(-3x^7-21x^6-63x^5-105x^4-105x^3-63x^2-21x-3\right)+\left(\frac{d^2}{dx^2}w(x)\right)^2\left(-15x^6-90x^5-219x^4-276x^3-189x^2-66x-9\right)+\left(\frac{d^2}{dx^2}w(x)\right)\left(\frac{d^3}{dx^3}w(x)\right)\left(-6x^5-30x^4-60x^3-60x^2-30x-6\right)+\left(\frac{d^2}{dx^2}w(x)\right)\left(-30x^4-120x^3-168x^2-96x-18\right)+\left(\frac{d^3}{dx^3}w(x)\right)^2\left(-2x^6-12x^5-30x^4-40x^3-30x^2-12x-2\right)+\left(\frac{d^3}{dx^3}w(x)\right)\left(-8x^5-40x^4-75x^3-65x^2-25x-3\right)-8x^4-32x^3-47x^2-30x-9=0$$

```
> simplify(eval(ADE4,w(x)=1/sin(x)+log(1+x)))
```

$$0=0 \tag{28}$$

```
> ADE5:=arithmeticDalg([ADE1,ADE2],[y(x),z(x)],w=y*z)
```

$$ADE5 := \left(-x^2-2x-3\right)w(x)^2+\left(-2x-2\right)w(x)\left(\frac{d}{dx}w(x)\right)+\left(4x^2+8x+6\right)w(x)\left(\frac{d^2}{dx^2}w(x)\right)+\left(1+x\right)\left(\frac{d^3}{dx^3}w(x)\right)w(x)+\left(-6x^2-12x-10\right)\left(\frac{d}{dx}w(x)\right)^2+\left(-3x-3\right)\left(\frac{d}{dx}w(x)\right)\left(\frac{d^2}{dx^2}w(x)\right)+\left(2x^2+4x+2\right)\left(\frac{d}{dx}w(x)\right)\left(\frac{d^3}{dx^3}w(x)\right)+\left(-3x^2-6x-3\right)\left(\frac{d^2}{dx^2}w(x)\right)^2=0 \tag{29}$$

```
> simplify(eval(ADE5,w(x)=1/sin(x)*log(1+x)))
```

$$0=0 \tag{30}$$

```
> ADE6:=arithmeticDalg([ADE1,ADE2],[y(x),z(x)],w=y/z)
```

$$ADE6 := \left(8x^4+32x^3+47x^2+30x+9\right)w(x)^2+w(x)\left(\frac{d}{dx}w(x)\right)\left(20x^3+60x^2+58x+18\right)+w(x)\left(\frac{d^2}{dx^2}w(x)\right)\left(8x^4+32x^3+56x^2+48x+18\right)+w(x)\left(\frac{d^3}{dx^3}w(x)\right)\left(8x^3+24x^2+25x+9\right)+\left(\frac{d}{dx}w(x)\right)^2\left(12x^4+48x^3+70x^2+44x+10\right)+\left(\frac{d}{dx}w(x)\right)\left(\frac{d^2}{dx^2}w(x)\right)\left(24x^3+72x^2+69x+21\right)+\left(\frac{d}{dx}w(x)\right)\left(\frac{d^3}{dx^3} \tag{31}$$

$$w(x)\Big)\left(16x^4+64x^3+94x^2+60x+14\right)+\left(\frac{d^2}{dx^2}w(x)\right)^2\left(9x^2+18x+9\right)+\Big(\frac{d^2}{dx^2}$$
$$w(x)\Big)\left(\frac{d^3}{dx^3}w(x)\right)\left(12x^3+36x^2+36x+12\right)+\Big(4x^4+16x^3+24x^2+16x$$
$$+4\Big)\left(\frac{d^3}{dx^3}w(x)\right)^2=0$$

```
> simplify(eval(ADE6,w(x)=sin(x)*log(1+x)))
```

$$0=0 \tag{32}$$

```
> ADE7:=unaryDalg(ADE1,y(x),w=(y+1)^3)
```

$$ADE7 := -27\,w(x)^2+\left(\frac{d}{dx}w(x)\right)^3\left(x^3+3x^2+3x+1\right)=0 \tag{33}$$

```
> simplify(eval(ADE7,w(x)=(log(1+x)^3)))
```

$$0=0 \tag{34}$$

Conclusion

The package is useful for practical computations with D-algebraic functions, i.e., solutions to algebraic differential equations. However, we point out that sometimes the running time can be very high due to elimination computations happening internally.